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F-distribution :- In probability thoery & statistics the F-distribution also known as Snedecor's F-distribution or the Fisher Snedecor's distribution is a continuous probability distribution that arises frequently as the null dist^{n} of a test statistic most notably in the analysis of variance (ANOVA) e.g. F-test

Def^{n}:
F-Statistic :- If X and Y are two indept. chi-square variate with ν_1 and ν_2 d.f. respectively then F-statistic is defined by

$$F = \frac{X/\nu_1}{Y/\nu_2}.$$

or F is defined as the ratio of two indept. chi-square variates divided by their respective degrees of freedom and it follows Snedecor's F-dist^{n} with (ν_1, ν_2) d.f. with probability function given by

$$f(F) = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \cdot \frac{F^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{\nu_1+\nu_2}{2}}} \quad ; \; 0 \le F < \infty$$

Remarks :- 1) The sampling dist^{n} of F-statistic does not involve any pop^{n} parameters and depends only on the degrees of freedom ν_1 and ν_2.

2) A statistic F following Snedecor's F-dist^{n} with (ν_1, ν_2) d.f. ~~will~~ denoted by

$F \sim F(\nu_1, \nu_2)$

Derivation of Snedecor's F- distⁿ:- Since X and Y are indept. chi-square variate with ν_1 and ν_2 d.f. respectively, their joint probability differential is given by

$$dF(x,y) = \left[\frac{1}{2^{\nu_1/2}\,\Gamma\nu_1/2} \exp(-x/2)\, x^{(\nu_1/2)-1}\, dx\right]$$

$$\times \left[\frac{1}{2^{\nu_2/2}\,\Gamma\nu_2/2} \exp(-y/2)\, y^{(\nu_2/2)-1}\, dy\right]$$

$$= \frac{1}{2^{(\nu_1+\nu_2)/2}\,\Gamma\nu_1/2\;\Gamma\nu_2/2} \exp\left[-(x+y)/2\right].$$

$$\times\ x^{(\nu_1/2)-1}\ y^{(\frac{\nu_2}{2})-1}\ dx\,dy \quad ; \; 0 \le (x,y) < \infty$$

Let us make the following transformation of variables

$$F = \frac{x/\nu_1}{y/\nu_2} \quad \text{and} \quad u = y, \text{ so that}$$

$$0 \le F < \infty$$
$$0 < u < \infty$$

$$x = \frac{\nu_1}{\nu_2} Fu = \frac{\nu_1}{\nu_2} Fu \quad \text{and} \quad y = u$$

Jacobian of transformation J is given by

$$J = \frac{\partial(x,y)}{\partial(F,u)} = \begin{vmatrix} \frac{\nu_1}{\nu_2}u & 0 \\ \frac{\nu_1}{\nu_2}F & 1 \end{vmatrix} = \frac{\nu_1 u}{\nu_2}$$

Thus the distribution of the transformed variable is

$$dG(F,u) = \frac{1 \times \exp\left\{-\frac{u}{2}\left(1+\frac{\nu_1}{\nu_2}F\right)\right\}}{2^{(\nu_1+\nu_2)/2}\,\Gamma\frac{\nu_1}{2}\,\Gamma\frac{\nu_2}{2}}$$

$$\times \left(\frac{\nu_1}{\nu_2}F u\right)^{\frac{\nu_1}{2}-1} u^{\frac{\nu_2}{2}-1} |J|\, du\, dF$$

$$= \frac{(\nu_1/\nu_2)^{\nu_1/2}}{2^{(\nu_1+\nu_2)/2}\,\Gamma\nu_1/2\,\Gamma\nu_2/2} \exp\left\{-\frac{u}{2}\left(1+\frac{\nu_1}{\nu_2}F\right)\right\}$$

$$\times\, u^{(\nu_1+\nu_2)/2-1}\, F^{(\nu_1/2)-1}\, du\, dF$$

$$0<u<\infty,\ 0\leq F<\infty$$

Integrating out u over the range 0 to ∞ the distn. of F becomes

$$g_1(F)\,dF = \frac{(\nu_1/\nu_2)^{\nu_1/2} F^{(\nu_1/2)-1}}{2^{(\nu_1+\nu_2)/2}\,\Gamma\frac{\nu_1}{2}\,\Gamma\frac{\nu_2}{2}}\, dF$$

$$\times \int_0^\infty \exp\left\{-\frac{u}{2}\left(1+\frac{\nu_1}{\nu_2}F\right)\right\} u^{(\nu_1+\nu_2)/2-1}\, du$$

$$g_1(F)\,dF = \frac{(\nu_1/\nu_2)^{\nu_1/2} (F)^{(\nu_1/2)-1}}{2^{(\nu_1+\nu_2)/2}\,\Gamma\frac{\nu_1}{2}\,\Gamma\nu_2/2} \cdot \frac{\Gamma\frac{\nu_1+\nu_2}{2}}{\left(\frac{1+\frac{\nu_1}{\nu_2}F}{2}\right)^{\frac{\nu_1+\nu_2}{2}}}$$

$$g_1(F)\,dF = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \frac{(F)^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{\nu_1+\nu_2}{2}}}\,dF \; ; \quad 0 \le F < \infty$$

which is required probability funⁿ of F-distⁿ with (ν_1, ν_2) d.f.

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$$F = \frac{x/\nu_1}{y/\nu_2}$$

$\therefore \frac{\nu_1}{\nu_2}F = \frac{x}{y}$, being the ratio of two indept. chi-square variate with ν_1 and ν_2 d.f respectively is a $\beta_2\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)$ variate. Hence the probability funⁿ of F is given by

$$dP(F) = \frac{1}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \frac{\left(\frac{\nu_1}{\nu_2}F\right)^{(\nu_1/2)-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{\nu_1+\nu_2}{2}}}\, d\left(\frac{\nu_1}{\nu_2}F\right)$$

$$= \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \cdot (F)^{(\nu_1/2)-1}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)\left[1+\frac{\nu_1}{\nu_2}F\right]^{\frac{\nu_1+\nu_2}{2}}}\, dF$$

$$0 \le F < \infty$$

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Constants of F-distn :—

The r^{th} moment about origin is given by

μ_r' (about origin) $= E(F^r)$

$$= \int_0^\infty F^r f(F)\, dF$$

$$= \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \int_0^\infty \frac{F^r (F)^{\nu_1/2 - 1}}{\left(1+\frac{\nu_1 F}{\nu_2}\right)^{\frac{\nu_1+\nu_2}{2}}} dF$$

To evaluate the integral, put

$\frac{\nu_1}{\nu_2} F = y$, so that $dF = \frac{\nu_2}{\nu_1} dy$

$$\mu_r' = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \int_0^\infty \frac{\left(\frac{\nu_2 y}{\nu_1}\right)^r \left(\frac{\nu_2}{\nu_1} y\right)^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2} \cdot \frac{\nu_2}{\nu_1} \cdot y\right)^{\frac{\nu_1+\nu_2}{2}}} \cdot \left(\frac{\nu_2}{\nu_1}\right) dy$$

$$= \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \left(\frac{\nu_2}{\nu_1}\right)^{r+\frac{\nu_1}{2}-1+1}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right) \cancel{(1+y)^{(\nu_1+\nu_2)/2}}} \int_0^\infty \frac{y^{\frac{\nu_1}{2}+r-1}}{(1+y)^{\frac{\nu_1+\nu_2}{2}}} dy$$

$$\mu_r' = \frac{\left(\frac{\nu_2}{\nu_1}\right)^r}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \int_0^\infty \frac{y^{\frac{\nu_1}{2}+r-1}}{(1+y)^{\frac{\nu_1}{2}+r+\frac{\nu_2}{2}-r}} dy$$

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$$\mu_r' = \frac{\left(\frac{\nu_2}{\nu_1}\right)^r}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \cdot B\left(\frac{\nu_1}{2}+r, \frac{\nu_2}{2}-r\right)$$

$; \nu_2 > 2r$

or $$\mu_r' = \frac{\left(\frac{\nu_2}{\nu_1}\right)^r}{\Gamma\frac{\nu_1}{2}\,\Gamma\frac{\nu_2}{2}} \cdot \frac{\Gamma\left(\frac{\nu_1}{2}+r\right)\Gamma\left(\frac{\nu_2}{2}-r\right)}{\cancel{\Gamma\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}\right)}} \cancel{\Gamma\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}\right)}$$

$$\mu_r' = \left(\frac{\nu_2}{\nu_1}\right)^r \frac{\Gamma\left(\frac{\nu_1}{2}+r\right)\Gamma\left(\frac{\nu_2}{2}-r\right)}{\Gamma\frac{\nu_1}{2}\,\Gamma\frac{\nu_2}{2}} \quad \text{--- } (*)$$

Now putting $r = 1$, we get

$$\mu_1' = \frac{\nu_2}{\nu_1} \cdot \frac{\Gamma\left(\frac{\nu_1}{2}+1\right)\Gamma\left(\frac{\nu_2}{2}-1\right)}{\Gamma\frac{\nu_1}{2}\,\Gamma\frac{\nu_2}{2}}$$

$$= \frac{\nu_2}{\nu_1} \cdot \frac{\frac{\nu_1}{2}\cdot\cancel{\Gamma\frac{\nu_1}{2}}}{\cancel{\Gamma\frac{\nu_1}{2}}} \cdot \frac{\cancel{\Gamma\left(\frac{\nu_2}{2}-1\right)}}{\left(\frac{\nu_2}{2}-1\right)\cancel{\Gamma\left(\frac{\nu_2}{2}-1\right)}}$$

$$\boxed{\because \Gamma r = (r-1)\,\Gamma(r-1)}$$

$$= \frac{\nu_2}{2} \cdot \frac{2}{(\nu_2-2)} = \frac{\nu_2}{(\nu_2-2)} \; ; \nu_2 > 2$$

is indept. of ν_1.

mean $\mu_1' = \frac{\nu_2}{\ldots}$

Putting $r = 2$ in (*)

$$\mu_2' = \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{\Gamma\frac{\nu_1}{2}+2 \; \Gamma\frac{\nu_2}{2}-2}{\Gamma\frac{\nu_1}{2} \; \Gamma\frac{\nu_2}{2}}$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{\left(\frac{\nu_1}{2}+1\right)\left(\frac{\nu_1}{2}\right)\cancel{\Gamma\frac{\nu_1}{2}} \; \cancel{\Gamma\frac{\nu_2}{2}-2}}{\cancel{\Gamma\frac{\nu_1}{2}} \left(\frac{\nu_2}{2}-1\right)\left(\frac{\nu_2}{2}-2\right) \cancel{\Gamma\frac{\nu_2}{2}-2}}$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{(\nu_1+2)}{\cancel{2}} \cdot \frac{\nu_1}{\cancel{2}} \cdot \frac{\cancel{2}}{(\nu_2-2)} \frac{\cancel{2}}{(\nu_2-4)}$$

$$= \frac{\nu_2^2}{\nu_1} \frac{(\nu_1+2)}{(\nu_2-2)(\nu_2-4)}, \quad \nu_2 > 4$$

then variance

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{1}{\nu_1} \frac{\nu_2^2 \, (\nu_1+2)}{(\nu_2-2)(\nu_2-4)} - \frac{\nu_2^2}{(\nu_2-2)^2}$$

$$\mu_2 = \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)(\nu_2-4)}, \quad \nu_2 > 4$$

is variance of F-statistic
similarly on putting $r=3$ and 4 in μ_r' we get μ_3' and μ_4' respectively. With help of moment about origin we find central moment